APPLICATIONS OF THE ATOMIC FUNCTIONS IN
THE ANTENNAS THEORY

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Abstract

The work is devoted to analysis of Atomic Functions (AF) applications to
principal problems of antennas theory. The analysis follows the result of the
solution of some applied problems, such as is to determine the field of antenna in
the far zone when the distribution of the current or the tangential components of the
electromagnetic field on the surface of antenna is known, and of synthesis is to
determine a current (or a field) in the antenna by means of radiation pattern in the
zone, and boundary value problems of the antenna theory is to determine the field in
antenna and surrounding area when excitation by outside sources and boundary
conditions are known.

Index Terms – Atomic functions, antenna theory, computational electromagnetic, boundary value
problem, antenna analysis and synthesis, approximation, current distribution, radiation pattern,
umerical method.

I – INTRODUCTION

Atomic Functions (AF) are widely used in different branches of computational physics
and also in digital interpretation of measurements of signals and in regeneration of images. For
the first time it is proposed to apply AF mathematical instrument to the antenna theory. The
advantages of this principally new approach in analysis and synthesis of the antenna were shown,
and also in boundary tasks of antenna theory. It was shown that AF in many cases are in the best
agreement with the physical content of the antenna concept, and take possibility to construct
simple and steady algorithms of calculation of their basic essential features.
A. Atomic functions

It is known that a large role in approximation theory and numerical methods is played by algebraic and trigonometric (exponential) polynomials. This universal instrument of approximation has an essential flaw in practice – the "non-finiteness" of classical polynomials. In numerical realisation, it is desirable to apply "local", i.e., finite functions, with carriers of small diameter, since in that case in the corresponding matrices the majority of elements are equal to 0, and the matrix is "rare" or even "ribbonlike", but at the corresponding matrices the majority of elements are equal to 0, and the matrix is "rare" or even "ribbonlike", but at the same volume of computer operating memory it is possible to use an approximating subspace of high dimensionality and thus receive precise approximations. The widely used local spline functions are not universal in the sense that to obtain an optimal approximation function with better smoothness, higher degree splines are needed. Thus, the classical algebraic and trigonometric polynomials are universal, but not local, and splines are local, but not universal. The question arises of the construction of spaces for functions that would be simultaneously local and universal.

The question arises: how are these functions constructed? We have in mind the fact that nonuniversal splines are conditioned by their already insufficient smoothness, which makes it necessary to consider infinitely differentiable finite functions. It is known that finite splines of the $N$-th degree are $N$-times folds of characteristic functions of intervals. These finite functions are infinitely repeated folds of characteristic functions of intervals. To obtain finite functions, the interval length should reach zero as quickly as possible. If $\phi(x)$ is a sought-for function and $\hat{\phi}(x)$ is its Fourier transform, then $\hat{\phi}(x)=\prod_{k=1}^{\infty} \frac{\sin \alpha_k t}{\alpha_k t}$; here $\alpha_k > 0$ and monotonically tends to zero, $\sum_{k=1}^{\infty} \alpha_k < \infty$. Out of all sequences $\alpha = |\alpha_k|$ tending monotonically to zero, we will pick those for which $\phi(x) = \phi_\alpha(x)$ possesses good approximation properties. It is evident that the simplest and most convenient can be the function $\phi(x)$ that corresponds to the geometric progression $\alpha_k=2^{-k}$. Such a choice of is at least logical. We will designate this function $up(x)$. Consequently,

$$up(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \prod_{k=1}^{\infty} \frac{\sin t2^{-k}}{t2^{-k}} dt$$

(1)

The original results for the theory of AF were obtained at the end of 1970 by V.L.Rvachev. The first publication appeared in 1971 in [1]. The problem of the existence of the simplest and, to date, the most important AF, $up(x)$, originating with V.L.Rvachev in 1967 and solved by him jointly with V.A.Rvachev, was a fundamentally new scientific result of the structural method suggested at the time by V.L.Rvachev for the solution of boundary value problems for equations in partial derivatives of mathematical physics. The functions $up(x)$ discussed here appeared under the following circumstances. In 1967, V.L.Rvachev put forth the following problem [2]. If $\phi(x)$ is a finite differentiable function, having one part rising and another falling (a "hump"), then its derivative $\phi'(x)$ will consist of a "hump" and a "hole". Does a function
φ(x) exist for which the "hump" and the "hole". Does a function φ(x) exist for which the "hump" and the "hole" of the derivative are similar to the "hump" of the function itself?

In the language of mathematics this means the following: does a finite solution, in which for definiteness we consider the carrier of φ(x) to be the segment [-1; 1], exist? In this manner, V.L.Rvachev and V.A.Rvachev proved in a 1971 work the existence and uniqueness of such a finite solution with an interval equal to 1, and this is the function \( \text{up}(x) \).

Thus, at the same time that classical algebraic and trigonometric equations with constant coefficients, the function \( \text{up}(x) \) and other analogous functions satisfy equations of the form

\[
L_y(x) = \lambda \sum_{k=1}^{m} c_k y(ax - bk),
\]

where \( a > 1 \), and \( L \) is a linear differential operator with constant coefficients. These equations are close to linear differential equations with constant coefficients in the sense that here the Fourier transformations operate effectively for them as well. In the theory of \( \text{AF} \) are studied \( \text{AF} \) and similar functions, spaces created by \( \text{AF} \), and also their application in approximation theory, mathematical analysis, the theory of differential functional equations in numerical methods, signal processing and image restoration, the statistical theory of knowledge of forms, boundary value problems of diffraction and scattering of electromagnetic waves, and problems of optimal control, analysis, and synthesis of antennas.

II - THE FUNCTION \( \text{up}(x) \)

The function \( \text{up}(x) \) [1-23] (Fig. 1) is defined from (1).

![Fig. 1. Atomic function \( \text{up}(x) \).](image)

It has compact support \([-1, 1]\) and is a solution of the functional-differential equation

\[
y'(x) = 2y(2x+1) - 2y(2x-1).
\]

Approximation of various classes of differentiable functions by expressions of the form
for fixed \( p \) as \( h \to 0 \) was investigated earlier in [3, 14]. In this note we study the order of approximation of differentiable functions by means of linear combination of translates of the function \( u^p(x) \) in the form

\[
\varphi(x) = \sum_{k=-\infty}^{\infty} c_k u^p\left(\frac{x}{h} - k2^{-p}\right)
\]

as \( p \to \infty \). It turns out that such approximations have a combination of properties not encountered with the classical trigonometric and algebraic polynomials and polynomial splines.

Let \( \{L_n\} \), \( n = 1, 2, \ldots \), be a sequence of subspaces of the space \( C[-1, 1] \) such that \( \dim L_n = n \), \( L_n \subset L_{n+1} \).

**Definition 1.** The sequence \( \{L_n\} \) is said to be **approximately universal** (a.u.) if for any natural number \( m \) and any \( f(x) \in C^m[-1, 1] \)

\[
\inf \|f - \varphi\|_{C[-1,1]} \leq C_m n^{-m} \left\| f^{(m)} \right\|_{C[-1,1]} \text{ for } n > n(m).
\]

**Definition 2.** The sequence \( \{L_n\} \) is said to be **sequence with local basis** (an l.b. sequence) if for any \( n \) there is a basis in \( L_n \) that consists of functions with compact supports whose diameters tend to 0 as \( n \to \infty \).

**Remark.** For subspaces of \( C[-1, 1] \) used in numerical methods both the property and the existence of a local basis are important, one reason being that in this case the matrices of the corresponding algebraic systems turn out to be "thINly populated".

It is well known, that spaces of algebraic (trigonometric, in the periodic case) polynomials are a.u., but they do not have local bases. The spaces of polynomial splines do not have local bases, consisting of the B-splines of Schoenberg, but they are not a.u. However, if we consider a sequence of spline subspaces \( \{L_n\} \) for which the degrees of the splines increase as \( n \to \infty \), then we do not have the following condition, which is important from a practical point of view

\[
L_n \subset L_{n+1}.
\]

**Theorem 1.** There is a sequence \( \{L_n\}, L_n \subset C[-1, 1] \), such that:

a) \( \dim L_n = n \),
b) \( L_n \subset L_{n+1} \),
c) \( L_n \) is a subspace of space of linear combination of the form (4), where \( p = \lfloor \log_2 n \rfloor \),
d) the sequence \( \{L_n\} \) is a.u.,
e) \( \{L_n\} \) is an l.b. sequence and the diameters of the supports of the functions in a local basis for \( L_n \) are equal to \( (p + 2) 2^p \).

**III - STATEMENT OF THE PROBLEM**

1) In the antennas theory the following questions are considered:
(i) problem of analysis is to determine the field of antenna in the far zone when the distribution of the current or the tangential components of the electromagnetic field on the surface of antenna is known,
(ii) problem of synthesis is to determine a current (or a field) in the antenna by means of radiation pattern in the far zone,
(iii) boundary value problem of the antenna theory – determination the field in antenna and in surrounding area when the activation by outside source and boundary conditions are given.

\( \text{AF} \) can be successfully applied to solve all this problems. Consider one of them.

2) Problem 1. This problem is reduced to solving vector linear equation

\[ LI = E, \]  \hspace{1cm} (4)

where \( L \) is linear operator produced by Green's function of the medium antenna placed, \( I \) is current distribution (electrical magnetic current or polarisation current) in antenna, \( E \) is field in far zone (usually considered a tangent components of electric or magnetic field to a sphere which radius is much more than the length of wave of light). To evaluate the advantages of applying \( \text{AF} \) to this problem lets consider one of the simplest following situations: analysis of the rectilinear antenna, which described by equation

\[ \int_{-l}^{l} E(u) \cdot A \cdot I(x) \cdot e^{-iux} dx = 0 \]  \hspace{1cm} (5)

where \( E(u) \) is complex amplitude of the field in the far zone that is proportional to the radiation pattern of the antenna, \( u = k \cos \theta \) (\( \theta \) is the angle of aspect, which is reading from normal to the antenna, \( A \) is complex constant which is not depended upon the angle of observation, \( 2l \) is the length of the antenna, \( I(x) = u(x) \) is distribution of the current (electric, magnetic or current of polarisation) in antenna that is placed along \( x \) axis of the Cartesian coordinate system. It is essential that \( I(x) \) should be finite function of \( x \) variable and should be differentiable, at last, two times (the low of the continuity of the current) and should be equal to zero on the end of the antenna. There in \( \text{AF} \) are very suitable to present function \( I(x) \). Actually, \( \text{AF} \) are finite and nonanalytic functions on the segment, smoothly approaching to zero on the end of the segment. For example all this conditions are satisfied by function \( u(x) \). Same is referred to atomic functions of the higher orders. As it is known, when the segment \([-l, l]\) is supplemented with infinity, the equation (2) can be considered as Fourier transform. Therefore when using \( \text{AF} \) relevant are their spectral properties.

The Fourier transform from \( \text{AF} u(x) \) i.e. the radiation pattern of allocation of a current \( I(x) = u(x) \) looks like

\[ F[u(x)] = \prod_{m=1}^{\infty} \frac{\sin u 2^{-m}}{u 2^{-m}} \]  \hspace{1cm} (6)

The Fourier transform of the \( \text{AF} \) belongs to class reactivity of entire functions of an exponential type of an extent not superior \( 2l \), descending on the material axis when \( u \to \infty \) exponentially.
In other words a set of radiation patterns which correspond to AF has property to concentrate energy in a range of real angles $-k \leq u \leq k$. Representation of a current (field) in a uniform linear array (among them, in aperture) with the help of the main AF [10, 12] ensures a radiation pattern with quick descending side lobes. It allows constructing antennas with optimal radiation patterns.

The approximation of composite distribution of a current $I(x)$ with the help of AF is possible by a partition of length of the antenna $2l$ on $2N$ of cuts of length $D=l/N$ and representation function $I(x)$ the following way:

$$I(x) = \sum_{n=-2^N}^{2^N} C(n) \cdot \text{up} \left( \frac{x/\Delta - n}{2^n} \right)$$

(7)

Thus, the radiation pattern will be introduced by the sum of the partial charts:
Thus, in the tasks of analysis of the theory of antennas a body of mathematics of AF allows to construct the constructive approach adequately representing analytical properties of these functions: finiteness, approaching to zero on the end, and concentration of a radiation energy in the area of physical observation angles (small reactance).

3) Problem 2. This problem is the inverse with the respect to the problem 1 and in the simplest case of the rectilinear antenna (6) it can be reduced to the solving of the integral equation of the 1st kind. As it was stated in the problem 1), the reduction of (6) to the form of bilateral Fourier transform requires complement of the \( I(x) \) function with zero on the rest of \( x \) axis and some complement of the \( E(x) \) function on the same interval without breaking the conditions of Wiener-Paley theorem. While using the AF for the representation of \( I(x) \), there are no difficulties with the provision of the finiteness of \( I(x) \) and with the progressing into the region of the superguided solutions, for avoiding of which one often refers to the methods of solution of incorrectly stated problems of mathematical physics. Consequently, the solution of the problem of antennas synthesis with the help of AF allows to suggest simple and stable algorithms for this class of problems.

4) Problem 3. The boundary value problems of antennas theory are, as a rule, solved with a Method of Moments (MoM), that is often called the generalised method of induced EMF. At the same time, from the boundary conditions there follows the equation \( Z \, I = U \), where \( Z \) is a linear (impedance) operator, \( I \) is a sought distribution of the current (electric, magnetic or polarisation) in the antenna, \( U \) is the given distribution of the primary (incident) field in the antenna. The important aspect of the MoM method is the choice of system of basic functions for decomposition of the sought current distribution. Formerly, for solving the problems of the antenna theory different systems of basic functions were used. The basic functions of subdomains upon the enough simplicity of the calculation of moments (reciprocal resistances or conductances) do not have the needed smoothness that often leads to errors upon solving the practical problems.

IV - CONCLUSION

The properties of the AF allows avoiding such errors, and as far as the partial radiation pattern are concentrated around zero, it guarantees the stability and exactness of calculation of field distribution in the far zone of the antenna, that is the first characteristic of any antenna by importance. The most attractive aspect is the application of AF in the theory of lattices of vibrator or slot antennas. Here the representation by AF is the most corresponding to the pattern of real distribution of the current in any transmitter, and consequently, it will yield the best convergence to the exact solution.
REFERENCES


