

Stochastic Analysis of the Laser Spectrum Considering the Phase Noise Effect

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Abstract

This paper presents a theoretical model for the estimation of the lineshape of lasers by the delayed self-heterodyne method, considering phase noise effects, based on [1]. Closed-forms expressions for the autocorrelation function and the power spectral density of the laser are derived, considering three cases: phase noise with low power, phase noise with medium power and phase noise with high power. The resulting expressions are valid for any optical phase noise probability distribution.

Index Terms

phase noise, laser spectrum, laser lineshape, delayed self-heterodyne method

I. INTRODUCTION

The increase in data traffic, especially originating from the Internet, which include applications such as video on demand, transfer of large databases from remote instruments used in telemedicine and scientific applications, and new applications for the network, such as parallel processing data that emerge from computational grids, motivated the use of wavelength-division multiplexing (WDM) as the basic technology for the optical communication systems.

The WDM concept was first published, in 1970, with the purpose of providing a better use of the available bandwidth in optical fibers using simultaneous transmission of multiple channels allocated to different wavelengths in the optical spectrum. The ITU standardized the WDM channel grid as multiple or fraction of 100 GHz. Typical WDM systems use up to 40 channels with spacing of 100 GHz or 80 channels with spacing of 50 GHz (Dense WDM) [2]. The number of channels may increase, specially if use is made of the coherent modulation technique and reduction of the channel spacing. The price is the susceptibility to a variety of sources of signal disruption.

The effect of the laser phase noise is the subject of the study developed in this paper. One of the widely accepted theories establishes that the width of the laser line can be influenced fluctuations

in the phase of the optical field. These fluctuations arise from spontaneous emission events, which discontinuously alter the phase and the intensity of the lasing field [3].

Analysis of the lineshape and measurement of the spectral linewidth of laser is essential for applications in coherent optical communication systems. This paper is mainly concerned with the estimation of laser lineshape. Theoretical closed-forms expressions, based on [1], for the autocorrelation function and the power spectral density are derived considering three cases: phase noise with low power, phase noise with medium power and phase noise with high power. The resulting expressions are valid for any optical phase noise probability distribution. It is assumed that the amplitude fluctuations of the source are negligible, and that the signal is detected by a square-law detector [4].

II. THE DELAYED SELF-HETERODYNE METHOD

Spectral lines emitted from lasers are not perfectly sharp. They have finite width whose 3 dB spectral spread is called linewidth (full width at the half maximum, FWHM).

The delayed self-heterodyne method is a procedure for high resolution measurement of a laser output spectrum, which eliminates the need for a separated local oscillator, a limitation of other methods. As seen in Fig. 1, the input beam from the laser under test is first divided by a beam splitter. One of the resulting optical beams passes through an external modulator, which shifts the frequency by an amount ω_s . The other beam, after a delay of t_d seconds, mixes with the first one, after which the combined beams are sensed by a quadratic photo-detector. When t_d is much larger than the coherence time of the laser, the two beams are uncorrelated.

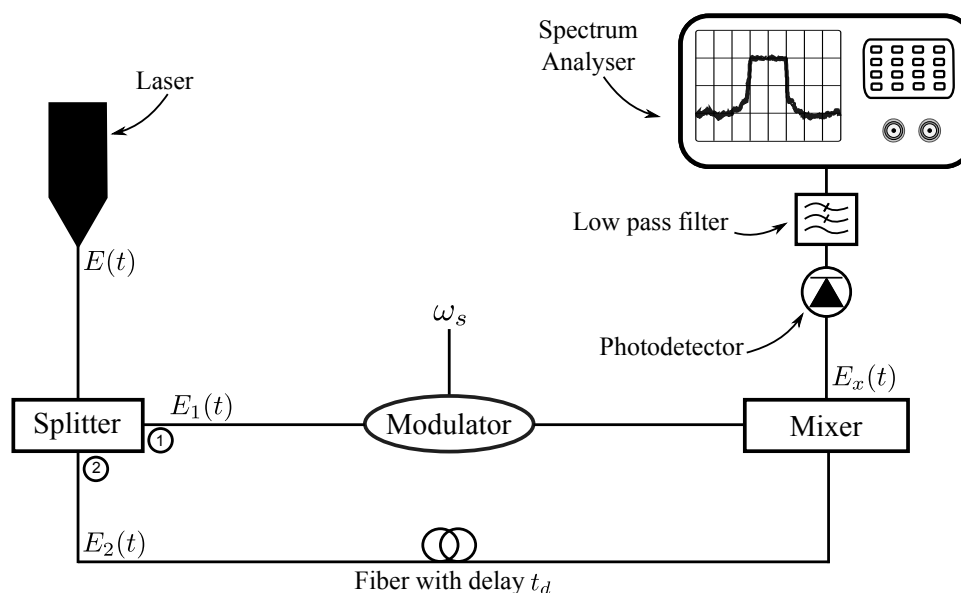


Fig. 1. Schematic setup for optical delayed self-heterodyne detection.

III. MODEL OF THE PHASE NOISE

The model presented in this section is based on delayed self-heterodyne method, described in Fig. 1.

The signal generated by a laser is an electromagnetic wave which can be represented by [3]

$$E(t) = E_0 \left[e^{j[\omega_0 t + \theta(t)]} + e^{-j[\omega_0 t + \theta(t)]} \right], \quad (1)$$

in which ω_0 is the oscillation frequency of the laser, E_0 is the amplitude and $\theta(t)$ is a zero mean random stationary process, defined by

$$\theta(t) = \int_{-\infty}^t \xi(t) dt, \quad (2)$$

in which $\xi(t)$ is the disturbance in the phase noise, related to the reciprocal of the laser output power [1], [5].

The photocurrent generated from the mixer, proportional to the received optical power, is expressed as [6]

$$\begin{aligned} i(t) &= \mathcal{R} \cdot |E_x(t)|^2 = \\ &= \mathcal{R} \cdot \left| E_1 e^{j[(\omega_0 + \omega_s)t + \theta(t)]} + e^{-j[(\omega_0 + \omega_s)t + \theta(t)]} + \right. \\ &\quad \left. + E_2 e^{j[\omega_0 t - \omega_0 t_d + \theta(t - t_d)]} + e^{-j[\omega_0 t - \omega_0 t_d + \theta(t - t_d)]} \right|^2. \end{aligned} \quad (3)$$

in which \mathcal{R} is the responsivity of the photodiode, $E_1(t)$ and $E_2(t)$ are the optical field in the branches 1 and 2 in the Fig. 1, respectively, and $E_x(t)$ is the optical field at the output of mixer.

Expanding the previous equation, and considering that terms with higher frequencies are removed by filtering, giving

$$i(t) = I_0 \left[e^{j[\omega_s t + \theta(t) - \theta(t - t_d)]} + e^{-j[\omega_s t + \theta(t) - \theta(t - t_d)]} \right], \quad (4)$$

in which $I_0 = 2\mathcal{R}E_1E_2e^{-j\omega_0 t_d}$.

The autocorrelation function of $i(t)$ can be expressed as

$$R_I(\tau) = \mathbb{E} \left[i(t)i(t + \tau) \right], \quad (5)$$

in which $\mathbb{E}[\cdot]$ is the expectancy operator.

After some mathematical manipulation, one obtains the expression for the autocorrelation

$$R_I(\tau) = \frac{I_0^2}{2} \cdot \mathbb{E} \left[\cos(\omega_s \tau + \theta(t + \tau) - \theta(t) - \theta(t + \tau - t_d) + \theta(t - t_d)) \right]. \quad (6)$$

In order to analyse the spectrum of the photocurrent subject to the phase noise, three cases are considered:

- a) phase noise with low variance;
- b) phase noise with medium variance;
- c) phase noise with high variance.

For each case, the autocorrelation and power spectral density are obtained.

A. Case I: Phase Noise with Low Variance

In this case, a high power output for the laser implies that variance of the phase noise is low.

The autocorrelation of $i(t)$ can be obtained from Eq. 6 by expanding the cosine

$$R_I(\tau) = \frac{I_0^2}{2} \cdot \cos(\omega_s \tau) \cdot \mathbb{E} \left[\cos \left(\theta(t + \tau) - \theta(t) - \theta(t + \tau - t_d) + \theta(t - t_d) \right) \right] \\ - \frac{I_0^2}{2} \cdot \sin(\omega_s \tau) \cdot \mathbb{E} \left[\sin \left(\theta(t + \tau) - \theta(t) - \theta(t + \tau - t_d) + \theta(t - t_d) \right) \right]. \quad (7)$$

It is possible to expand the sine and cosine of Eq. 7 in Taylor series, neglecting the higher order terms, to obtain

$$R_I(\tau) = \frac{I_0^2}{2} \cdot \cos(\omega_s \tau) \cdot \left(1 - \mathbb{E} \left[\frac{\varphi^2}{2} \right] \right) - \frac{I_0^2}{2} \cdot \sin(\omega_s \tau) \cdot \mathbb{E}[\varphi], \quad (8)$$

in which $\varphi = \theta(t + \tau) - \theta(t) - \theta(t + \tau - t_d) + \theta(t - t_d)$ and $\cos(\omega_s \tau)$ and $\sin(\omega_s \tau)$ are deterministic.

Since $\theta(t)$ is a zero mean stationary process, it follows that

$$R_I(\tau) = \frac{I_0^2}{2} \cdot \cos(\omega_s \tau) \cdot [1 - 2R_\theta(0) + 2R_\theta(\tau)] \\ - \frac{I_0^2}{2} \cdot \cos(\omega_s \tau) \cdot [R_\theta(\tau - t_d) + R_\theta(\tau + t_d)], \quad (9)$$

in which $R_\theta(\tau)$ is the autocorrelation of the phase noise $\theta(t)$.

The power spectral density of $i(t)$ is obtained by the use of Wiener-Kintchine theorem, applying the Fourier transform to Eq. 9 [7]. Thereby, considering positive frequencies only,

$$S_I(\omega) = \frac{I_0^2}{2} [1 - 2R_\theta(0)] [\delta(\omega - \omega_s)] \\ + \frac{I_0^2}{4} \cdot S_\theta(\omega - \omega_s) \left[2 - \cos[(\omega - \omega_s)t_d] \right], \quad (10)$$

in which $S_\theta(\omega)$ represents the spectrum of the phase noise and $\delta(\omega)$ is the impulse function.

B. Case II: Phase Noise with Medium Variance

Increasing the power of phase noise implies, considering additional terms from the Taylor expansion of Eq. 7, and considering that the bandwidth of the carrier does not exceed four times the bandwidth of the phase noise, it is possible to neglect all terms of the expansion of order higher than four. Therefore,

$$R_I(\tau) = \frac{I_0^2}{4} \cdot \cos(\omega_s \tau) \cdot \left(1 - \mathbb{E}[\varphi^2] + \mathbb{E} \left[\frac{\varphi^4}{12} \right] \right), \quad (11)$$

in which $\varphi = \theta(t + \tau) - \theta(t) - \theta(t + \tau - t_d) + \theta(t - t_d)$, $\cos(\omega_s \tau)$ and $\sin(\omega_s \tau)$ are deterministic and all terms of the sine expansion vanish because they are joint moments of odd order [8].

According to [7], for Gaussian variables,

$$\mathbb{E}[x^n] = \begin{cases} (n-1)\sigma^n, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd.} \end{cases} \quad (12)$$

Applying 12 in the Eq. 11, it follows that

$$\begin{aligned} R_I(\tau) = & \frac{I_0^2}{4} \cdot \cos(\omega_s \tau) \cdot \left[1 - 4R_\theta(0) + 4R_\theta^2(0) \right] \\ & + \frac{I_0^2}{4} \cdot \cos(\omega_s \tau) \cdot 4R_\theta(0) \left[R_\theta(\tau - t_d) + R_\theta(\tau + t_d) - 2R_\theta(\tau) \right] \\ & - \frac{I_0^2}{4} \cdot \cos(\omega_s \tau) \cdot 4R_\theta(\tau) \left[R_\theta(\tau - t_d) + R_\theta(\tau + t_d) - 1 \right] \\ & + \frac{I_0^2}{4} \cdot \cos(\omega_s \tau) \cdot R_\theta(\tau - t_d) \left[R_\theta(\tau - t_d) + 3R_\theta(\tau + t_d) - 2 \right] \\ & + \frac{I_0^2}{4} \cdot \cos(\omega_s \tau) \cdot R_\theta(\tau + t_d) \left[R_\theta(\tau + t_d) - 2 \right] \end{aligned} \quad (13)$$

and the Fourier transform of this expression is

$$\begin{aligned} S_I(\omega) = & \frac{I_0^2}{4} \cdot \left[1 - 4R_\theta(0) + 4R_\theta^2(0) \right] \cdot [\delta(\omega - \omega_s)] \\ & + I_0^2 \cdot R_\theta(0) \cdot S_\theta(\omega - \omega_s) \cdot \left[\cos[(\omega - \omega_s)t_d] - 2 \right] \\ & - I_0^2 \cdot S_\theta(\omega - \omega_s) \cdot \left[S_\theta(\omega) * \cos[(\omega - \omega_s)t_d] + 1 \right] \\ & + \frac{I_0^2}{4} \cdot \left[S_\theta(\omega - \omega_s) \cdot e^{-j(\omega - \omega_s)t_d} \right] * \left[S_\theta(\omega) \cdot \cos(\omega t_d) - 1 \right] \\ & + \frac{I_0^2}{4} \cdot \left[S_\theta(\omega - \omega_s) \cdot e^{j(\omega - \omega_s)t_d} \right] * \left[S_\theta(\omega) \cdot e^{-j(\omega)t_d} + 2 \right], \end{aligned} \quad (14)$$

in which the symbol * represents the convolution operation.

C. Case III: Phase Noise with High Variance

In this case, a low output power of the laser is considered, which implies a high variance for the disturbance. The representation of sine and cosine terms in a Taylor series is not feasible, because it creates difficulties in view of the spectrum.

For this case, one uses a variation of Woodward's theorem. This theorem establishes that, for frequency modulation, the power spectral density of the modulated carrier turns into the probability density function of the modulating signal [9].

For the phase noise with high power case, it is more interesting to use Euler's formula and rewrite the Eq. 6 as

$$R_I(\tau) = \frac{I_0^2}{4} \cdot e^{j\omega_s\tau} \cdot \mathbb{E} \left[e^{j(\theta(t+\tau) - \theta(t) - \theta(t+\tau-t_d) + \theta(t-t_d))} \right] \\ + \frac{I_0^2}{4} \cdot e^{-j\omega_s\tau} \cdot \mathbb{E} \left[e^{-j(\overbrace{\theta(t+\tau) - \theta(t) - \theta(t+\tau-t_d) + \theta(t-t_d)}^{\textcircled{B}})} \right]. \quad (15)$$

In order to simplify the \textcircled{B} term, a possible approach is to use an approximation with a second order mean square estimator [10]. Thereby, the difference $\Delta\theta$ can be written as

$$\Delta\theta = [\theta(t+\tau) + \theta(t-t_d)] - \alpha[\theta(t) + \theta(t+\tau-t_d)] - \beta[\theta'(t) + \theta'(t+\tau-t_d)], \quad (16)$$

in which $\theta'(t) = d\theta(t)/dt$.

The mean square error is given by

$$\epsilon(t, \alpha, \beta) = \mathbb{E} \left[\left([\theta(t+\tau) + \theta(t-t_d)] - \alpha[\theta(t) + \theta(t+\tau-t_d)] \right. \right. \\ \left. \left. - \beta[\theta'(t) + \theta'(t+\tau-t_d)] \right)^2 \right]. \quad (17)$$

The minimum error occurs when the partial derivatives with relation to α and β converge to zero. Thus,

$$\frac{\partial \epsilon(t, \alpha, \beta)}{\partial \alpha} = -4R_\theta(\tau) + 4\alpha[R_\theta(0) + R_\theta(\tau-t_d)] + 4\beta R_{\theta\theta'}(0) = 0 \quad (18a)$$

$$\frac{\partial \epsilon(t, \alpha, \beta)}{\partial \beta} = -4R_{\theta'\theta}(\tau) + 4\beta[R_{\theta'}(0) + R_{\theta'}(\tau-t_d)] + 4\alpha R_{\theta\theta'}(0) = 0. \quad (18b)$$

Rearranging the Eq.18a and 18b, and considering $R_{\theta\theta'}(0) = 0$, because the autocorrelation has a maximum at the origin [10], one obtains

$$\alpha = \frac{R_\theta(\tau)}{R_\theta(0) + R_\theta(\tau-t_d)} \quad \text{and} \quad (19a)$$

$$\beta = \frac{R_{\theta'\theta}(\tau)}{R_{\theta'}(0) + R_{\theta'}(\tau-t_d)}. \quad (19b)$$

Considering that the phase noise is slowly varying, as compared to the spectral frequency of the carrier, it can be considered that $\tau \rightarrow 0$, therefore

$$R_{\theta}(\tau) \approx R_{\theta}(0), \quad (20a)$$

$$R_{\theta}(\tau - t_d) \approx 0, \quad (20b)$$

in which the delay t_d is larger than the coherence time of laser and, thus,

$$\alpha = \frac{R_{\theta}(\tau)}{R_{\theta}(0) + R_{\theta}(\tau - t_d)} = \frac{R_{\theta}(0)}{R_{\theta}(0)} = 1. \quad (21)$$

From the properties of the autocorrelation, $R_{\theta'}(\tau) = -R_{\theta}''(\tau)$ [7]. Expanding the derivatives of the autocorrelation in a Taylor series, and recalling that the autocorrelation has a maximum at the origin,

$$R_{\theta'}(\tau) = -\tau R_{\theta}''(0). \quad (22)$$

As $\tau \rightarrow 0$ and, according to [7], $R_{x'}(\tau) = -R_{x}''(\tau)$, therefore

$$R_{\theta'}(0) + R_{\theta'}(\tau - t_d) \approx -R_{\theta}''(0) - R_{\theta}''(-t_d) = -R_{\theta}''(0). \quad (23)$$

Thus,

$$\beta = \frac{R_{\theta'}(\tau)}{R_{\theta'}(0) + R_{\theta'}(\tau - t_d)} = \frac{-\tau R_{\theta}''(0)}{-R_{\theta}''(0)} = \tau \quad (24)$$

Finally, the \textcircled{B} term of the Eq. (15) can be approximated by

$$\begin{aligned} \textcircled{B} &= \theta(t + \tau) - \theta(t) - \theta(t + \tau - t_d) + \theta(t - t_d) \\ &\approx \tau [\theta'(t) + \theta'(t + \tau - t_d)] \approx \tau [\theta'(t) + \theta'(t - t_d)], \end{aligned} \quad (25)$$

considering $\tau \rightarrow 0$.

Use of the linear mean square estimator in Eq. 15 then gives [10]

$$\begin{aligned} R_I(\tau) &= \frac{I_0^2}{4} \cdot e^{j\omega_s\tau} \cdot \mathbb{E} \left[e^{j(\tau\theta'(t) + \tau\theta'(t - t_d))} \right] \\ &\quad + \frac{I_0^2}{4} \cdot e^{-j\omega_s\tau} \cdot \mathbb{E} \left[e^{-j(\tau\theta'(t) + \tau\theta'(t - t_d))} \right] \end{aligned} \quad (26)$$

Owing to the lack of correlation between $\theta'(t)$ and $\theta'(t - t_d)$, and the stationarity of the phase noise, one obtains

$$R_I(\tau) = \frac{I_0^2}{4} \cdot e^{j\omega_s\tau} \cdot \mathbb{E}^2 \left[e^{j\tau\theta'(t)} \right] + \frac{I_0^2}{4} \cdot e^{-j\omega_s\tau} \cdot \mathbb{E}^2 \left[e^{-j\tau\theta'(t)} \right]. \quad (27)$$

The characteristic function of a stochastic process given by

$$\mathbb{E}[e^{j\tau x}] = \int_{-\infty}^{\infty} p_X(x) e^{j\tau x} dx = \Phi_X(\tau), \quad (28)$$

in which $\Phi_X(\tau)$ represents the characteristic function and $p_X(x)$ is the probability density function of the process, Eq. 27 can be written as

$$R_I(\tau) = \frac{I_0^2}{4} \cdot e^{j\omega_s \tau} \cdot \Phi_\theta^2(\tau) + \frac{I_0^2}{4} \cdot e^{-j\omega_s \tau} \cdot \Phi_\theta^2(\tau). \quad (29)$$

in which $\Phi_\theta(\tau)$ is the characteristic function of the phase noise.

Calculating the Fourier transform of Eq.29, it follows that

$$S_I(\omega) = \frac{I_0^2}{2} [p_\theta(\omega - \omega_s) * p_\theta(\omega - \omega_s)], \quad (30)$$

in which $p_\theta(\cdot)$ is the probability density function of the phase noise.

IV. DISCUSSION OF THE RESULTS

To validate the model developed, the power spectrum density for the three cases studied are plotted in Fig. 2, considering a Gaussian phase noise distribution.

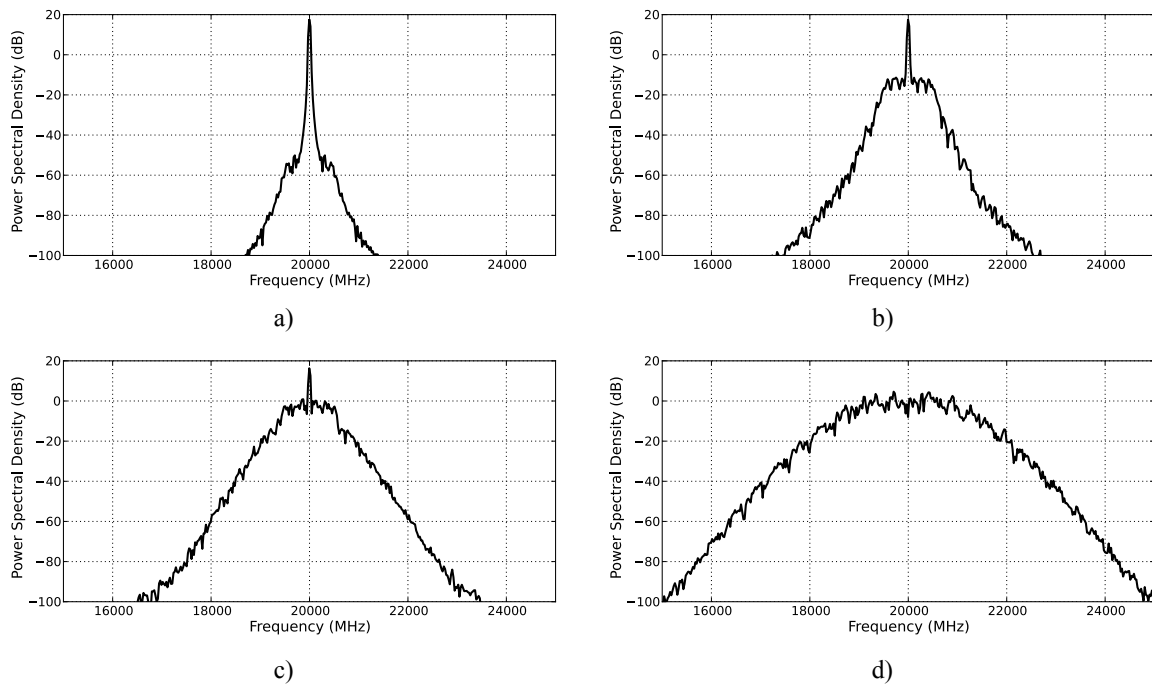


Fig. 2. Lineshape of the laser for: a) phase noise with low power; b) and c) phase noise with medium power; d) phase noise with high power.

In this figure, a) represents the power spectrum density of case I, low power phase noise. The plots b) and c) relate to case II, medium power phase noise, and plot d) relates to case III, high power phase noise.

Eq. 10 is in agreement with several experimental results from the literature [4], [11]. In this case, the laser lineshape approach a $1/f^2$ curve, which indicates a Lorentzian spectrum.

In Eq. 14, the convolution operation is responsible for broadening the spectrum, as can be seen by comparing Fig. 2 a), case I, and Fig. 2 b) and c), as the power increases, in case II.

In case III the lineshape converges to the probability density function of the phase noise, as shown the Eq. 30. This means that for a Gaussian phase noise distribution, the laser lineshape will also be Gaussian, when the disturbance attains a high level. The result is in agreement with the experimental results of [12] and [13] and explain the Gaussian lineshape encountered in [14], in relation to the linewidth floor. It also shows that, owing to the convolution operation, the linewidth is twice that expected.

V. CONCLUSION

The main objective of this paper is to model the laser spectrum considering phase noise effects. A theoretical formulation, using stochastic process, for the laser spectral lineshape, using the delayed self-heterodyne method, produced generalized closed-forms expressions for the autocorrelation function and the power spectral density considering three cases: phase noise with low, medium and high power. The general aspect of the derived expressions comes from the fact they are valid for any phase noise probability distribution. The theoretical results agree with experimental results available in the literature.

REFERENCES

- [1] M. S. de Alencar, "Analysis of the Spectral Lineshape of Semiconductor Lasers," in *Simpósio Brasileiro de Telecomunicações*, 1991.
- [2] K. S. e. G. S. Rajiv Ramaswami, *Optical Networks: A Practical Perspective*, 3rd ed. Elsevier, 2009.
- [3] C. H. Henry, "Theory of the Linewidth of Semiconductor Lasers," *IEEE Journal of Quantum Electronics*, vol. 18, no. 2, pp. 259–264, 1982.
- [4] B. Moslehi, "Noise Power Spectra of Optical Two-Beam Interferometers Induced by the Laser Phase Noise," *Journal of Lightwave Technology*, vol. 4, no. 11, pp. 1704–1710, 1986.
- [5] C. H. Henry, "Phase Noise in Semiconductor Lasers," *Journal of Lightwave Technology*, vol. 4, no. 3, pp. 298–311, 1986.
- [6] G. P. Agrawal, *Fiber-Optic Communication Systems*. Wiley-Interscience, 2002.
- [7] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*, 4th ed. McGraw-Hill, 2002.
- [8] A. Papoulis, "Random Modulation: A Review," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 31, no. 1, pp. 96–105, 1983.
- [9] N. M. Blachman e G. A. McAlpine, "The Spectrum of a High-Index FM Waveform: Woodward's Theorem Revisited," *IEEE Transactions on Communications Technology*, vol. 17, no. 2, 1969.
- [10] M. S. de Alencar and V. C. da Rocha, *Communication Systems*. Springer, 2005.
- [11] R. W. Tkach and A. R. Chraplyvy, "Phase Noise and Linewidth in an InGaAsP DFB Laser," *Journal of Lightwave Technology*, vol. 4, no. 11, pp. 1711–1716, 1986.
- [12] K. Kikuchi, "Precise Estimation of Linewidth Reduction in Wavelength-Detuned DFB Semiconductor Lasers," *Electronics Letters*, vol. 24, no. 2, pp. 80–81, 1988.
- [13] A. Chraplyvy, K. Liou, R. Tkach, G. Eisenstein, Y. Jhee, T. Koch, P. Anthony, and U. Chakrabarti, "Simple Narrow-Linewidth 1.5 μm InGaAsP DFB External-Cavity Laser," *Electronics Letters*, vol. 22, no. 2, pp. 88–90, 1986.
- [14] H. Yasaka, M. Fukuda, and T. Ikegami, "Current Tailoring for Lowering Linewidth Floor," *Electronics Letters*, vol. 24, no. 12, pp. 760–762, 1988.